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Propagator for narrow potential barriers

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Abstract
The Schrödinger equation in integral form is applied to the one-dimensional scattering problem in the case of a narrow potential barrier. Since the kernel can be considered, in a first approximation, separable, an explicit expression for the propagator is found by means of the complementary error function. The problem of a particle confined in a half-space and interacting with a narrow potential barrier is also considered and solved in an approximate way.

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1. Introduction

The spacetime propagator can be considered the most important object in quantum physics; it governs the time evolution of a dynamical state and naturally enters any kind of time-dependent problem [1]. We would like to point out that it is more fundamental than the wavefunction itself, since it is characteristic of the physical system and does not depend on initial conditions. Its calculation is more difficult, however; suffice it to consider the square potential: the wavefunctions are well known, but the propagator cannot be expressed in a simple form. On the other hand, knowledge of the propagator helps to give insight into the physics of a quantum system. For example, let us think of the tunnelling time for a potential barrier: in a basic paper on this argument, it is shown that a satisfactory definition of the tunnelling time can be given just by means of the propagator [2]; considering coupling effects in quantum tunnelling, it makes it possible to go beyond the perturbative expansion [3]; studying the interaction of a system with a thermal bath, it accounts in a simple way for the oscillatory degrees of freedom, thus leading to the concept of effective action [4].

The non-relativistic quantum mechanical propagator can be expressed in several ways. The most widely known is the 'spectral decomposition' method, but also the path-integral approach is often used. We refer to the literature for a complete discussion on this subject [5], and limit ourselves to a brief 'excursus'.

First, every kind of quadratic Hamiltonian admits an explicit solution, due to the fact that, in this case, the semiclassical approach is exact [6, 7]. A closed-form solution is also known for a variety of potentials [7, 8], but for the simplest one, the piecewise-constant potential, the result is very involute or even missing in the literature. The case of the potential step has been completely solved by de Carvalho [9], which used the path-decomposition (PDX) technique [10], in order to express the result as an integral of simpler propagators. The extension to the square barrier is not, however, a simple task.

The Green function (or energy propagator) for an arbitrary barrier was treated by de Aguiar [11], in a paper where the calculations are scarcely detailed and the possibility of finding the propagator by a Fourier transform is only outlined. Finally, we cite a paper of Barut and Duru [12], where the phase space path integral is used; by a canonical transformation the original Hamiltonian is removed, so reducing the propagator to a very simple form and obtaining a surprising result, in view of its compactness and generality. A careful inspection, however, shows that it is incorrect, due to the fact that the phase space path integral is not invariant under canonical transformations. It is not the intention here to expand on this subtle matter: the reader will find simple and well-written considerations in Shulman's book [13]. In subsequent years, to our knowledge, no important contribution has been added to the subject. We show in this paper that the integral Schrödinger equation allows us to find an interesting approximate expression of the propagator for narrow barriers; in fact, in this case a Fredholm equation of the second kind with separable kernel is obtained, and the solution easily follows.

2. The integral equation for narrow barriers

Let H_0 be an Hamiltonian for which the propagator G_0 is known, and V a general potential. The Schrödinger equation for the system with Hamiltonian $H = H_0 + V$ is

$$\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\psi(t)\rangle = H|\psi(t)\rangle \tag{1}$$

where $|\psi(t)\rangle$ is the vector representing the dynamical state of our system. Upon differentiating with respect to τ the expression

$$\left[\exp -\frac{\mathrm{i}}{\hbar} H_0(t-\tau)\right] |\psi(\tau)\rangle, \tag{2}$$

with $\tau < t$, and using equation (1), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H_0(t-\tau)}|\psi(\tau)\rangle\right] = -\frac{\mathrm{i}}{\hbar}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H_0(t-\tau)} V|\psi(\tau)\rangle. \tag{3}$$

By integration, it follows at once that

$$|\psi(t)\rangle = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H_0 t}|\psi(0)\rangle - \frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}\tau\, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H_0(t-\tau)} V|\psi(\tau)\rangle \tag{4}$$

that is the Schrödinger equation in integral form. Let us suppose that H_0 corresponds to the free particle; using the representation where the position variables are diagonal [14] (namely, passing from the state vectors $|\psi(t)\rangle$ to the wavefunctions $\psi(x,t)$), this equation becomes

$$\psi(x,t) = \int_{-\infty}^{\infty} \mathrm{d}\xi\, \psi_0(\xi) G_0(x,t;\xi) - \frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}\xi\, V(\xi,\tau)\psi(\xi,\tau) G_0(x,t-\tau;\xi) \tag{5}$$

where

$$G_0(x,t;\xi) = \langle x|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H_0 t}|\xi\rangle = \sqrt{\frac{m}{2\pi \mathrm{i}\hbar t}} \exp\left[\mathrm{i}\frac{m}{2\hbar}\frac{(x-\xi)^2}{t}\right] \tag{6}$$

is the propagator for the free particle and ψ_0 is the wavefunction for $t = 0$. The Wick rotation to imaginary time [3, 10] leads to

$$\psi(x,t) = \sqrt{\frac{m}{2\pi\hbar}}\left[\int_{-\infty}^{\infty} \mathrm{d}\xi\ \psi_0(\xi)\frac{\mathrm{e}^{-\frac{m}{2\hbar}\frac{(x-\xi)^2}{t}}}{\sqrt{t}} - \frac{1}{\hbar}\int_0^t \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}\xi\ V(\xi,\tau)\psi(\xi,\tau)\frac{\mathrm{e}^{-\frac{m}{2\hbar}\frac{(x-\xi)^2}{t-\tau}}}{\sqrt{t-\tau}}\right]. \tag{7}$$

By considering a potential barrier of the form (θ is the step function)

$$V(x,t) = V(x)[\theta(x) - \theta(x-a)] \tag{8}$$

and using the Laplace transformation: $\mathcal{L}\{\psi(x,t)\} = \int_0^\infty \mathrm{d}t\ \psi(x,t)\exp(-st) = \psi(x,s)$, we obtain [16]

$$\psi(x,s) = \sqrt{\frac{m}{2\hbar}}\left[\int_{-\infty}^{\infty} \mathrm{d}\xi\ \psi_0(\xi)\frac{\mathrm{e}^{-\sqrt{\frac{2m}{\hbar}}|x-\xi|\sqrt{s}}}{\sqrt{s}} - \frac{1}{\hbar}\int_0^a \mathrm{d}\xi\ V(\xi)\psi(\xi,s)\frac{\mathrm{e}^{-\sqrt{\frac{2m}{\hbar}}|x-\xi|\sqrt{s}}}{\sqrt{s}}\right]. \tag{9}$$

With the abbreviations

$$\lambda = \frac{1}{\hbar}\sqrt{\frac{m}{2\hbar}}\frac{1}{\sqrt{s}}, \qquad k = \sqrt{\frac{2m}{\hbar}}\sqrt{s}, \qquad \phi(x) = \sqrt{\frac{m}{2\hbar}}\int_{-\infty}^{\infty} \mathrm{d}\xi\ \psi_0(\xi)\frac{\mathrm{e}^{-\sqrt{\frac{2m}{\hbar}}|x-\xi|\sqrt{s}}}{\sqrt{s}} \tag{10}$$

this equation can be written in short as

$$\psi(x) + \lambda\int_0^a \mathrm{d}\xi\ \mathrm{e}^{-k|x-\xi|}V(\xi)\psi(\xi) = \phi(x) \tag{11}$$

where the variable s, considered as a parameter, is omitted, since we are now mainly interested in the space variable x.

If $x \notin [0, a]$, the kernel is separable: it is, namely, a Pincherle–Goursat (or 'degenerate') kernel [15]. An approximate solution can be found using the fact that, if in the potential (8) the range a is very small, the kernel in equation (11) can be considered to be nearly everywhere separable. To see this in detail, let us consider for every $\notin [0, a]$ the equation

$$\psi(x) + \lambda\int_0^a \mathrm{d}\xi\ f_1(x)f_2(\xi)\psi(\xi) = \phi(x) \tag{12}$$

with

$$\int_0^a \mathrm{d}x\ f_1(x)f_2(x) = A. \tag{13}$$

The setting

$$y = \int_0^a \mathrm{d}\xi\ f_2(\xi)\psi(\xi) \tag{14}$$

leads to

$$\psi(x) = \phi(x) - \lambda y f_1(x) \tag{15}$$

that is nothing but a shorthand of equation (12); now, let us perform the approximation that this equation holds for every x. In other words, we simplify the problem by supposing that the kernel is separable also for $x \in [0, a]$. Therefore, it is possible to substitute equation (15) into (14):

$$y = \int_0^a \mathrm{d}\xi\ f_2(\xi)[\phi(\xi) - \lambda y f_1(\xi)] = \int_0^a \mathrm{d}\xi\ f_2(\xi)\phi(\xi) - \lambda A y. \tag{16}$$

Therefore

$$y = \frac{1}{1+\lambda A}\int_0^a \mathrm{d}\xi\, f_2(\xi)\phi(\xi) \qquad \text{and} \qquad \psi(x) = \phi(x) - \frac{\lambda}{1+\lambda A} f_1(x)\int_0^a \mathrm{d}\xi\, \phi(\xi) f_2(\xi). \tag{17}$$

Let us apply this result to equation (11), where the kernel $V(\xi)\exp(-k|x-\xi|)$ is separable only when $x \notin [0, a]$. Choosing

$$\begin{aligned} &f_1(x) = \mathrm{e}^{-kx}, \qquad f_2(\xi) = V(\xi)\,\mathrm{e}^{k\xi} \qquad \text{for} \quad x > a, \\ &f_1(x) = \mathrm{e}^{kx}, \qquad f_2(\xi) = V(\xi)\,\mathrm{e}^{-k\xi} \qquad \text{for} \quad x < 0, \end{aligned} \tag{18}$$

the final result is

$$\psi(x) = \phi(x) - \frac{\lambda}{1+\lambda\gamma}\int_0^a \mathrm{d}\xi\, \mathrm{e}^{-k|x-\xi|} V(\xi)\phi(\xi), \qquad \text{with} \qquad \gamma = \int_0^a \mathrm{d}x\, V(x). \tag{19}$$

We point out that the approximation, in this procedure, lies in the substitution of equation (15) into (14); and the error is as lower, as shorter is the range a.

3. An expression for the short-range potential

Starting from the expansion in a series of derivatives of the Dirac delta function $\delta(x)$:

$$\frac{1}{\sqrt{\pi\alpha}}\exp\left(-\frac{x^2}{\alpha}\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\alpha}{4}\right)^n \delta^{(2n)}(x) \tag{20}$$

that can be proved in the simplest way by taking the Fourier transform [18] of the Taylor series

$$\exp\left(-\frac{x^2}{4}\alpha\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{\alpha}{4}\right)^n x^{2n}, \tag{21}$$

with reference to the potential (8), this form for $V(x)$ can be considered:

$$V(x) = U(x)\frac{1}{\sqrt{\pi\alpha}}\exp\left[-\frac{(x-\bar{a})^2}{\alpha}\right] \approx U(x)\left[\delta(x-\bar{a}) + \frac{\alpha}{4}\delta^{(2)}(x-\bar{a})\right], \tag{22}$$

with $U(x)$ smooth, $\bar{a}$ inside the interval $[0, a]$, and for small values of α. It easily allows for an approximate analytical solution; in this case

$$\gamma = \int_0^a \mathrm{d}x\, V(x) = \left[U(x) + \frac{\alpha}{4}U^{(2)}(x)\right]_{x=\bar{a}} \tag{23}$$

where a well-known formula is used [18]:

$$\int_{-\infty}^{\infty}\mathrm{d}x\, \delta^{(n)}(x) f(x) = (-1)^n f^{(n)}(0). \tag{24}$$

4. The explicit form of the propagator

Let us suppose that ψ_0 is different from zero on the left-hand side of the barrier only; by introducing into equation (19) the explicit expressions of $\phi(x)$, λ and k (see equation (10)) and restoring the variable s, we have

$$\psi(x,s) = \frac{c}{2}\int \mathrm{d}\eta \left[\frac{\mathrm{e}^{-c|x-\eta|\sqrt{s}}}{\sqrt{s}} - \frac{c}{2\hbar}\int_0^a \mathrm{d}\xi\, V(\xi)\frac{\mathrm{e}^{-c(|x-\xi|+\xi-\eta)\sqrt{s}}}{\sqrt{s}(\sqrt{s}+\beta)}\right]\psi_0(\eta) \tag{25}$$

where

$$c = \sqrt{\frac{2m}{\hbar}}, \qquad \beta = \frac{c}{2}\frac{\gamma}{\hbar} \tag{26}$$

and $V(x)$ is given by equation (22).

Performing the inversion $\mathcal{L}^{-1}$ to obtain $\psi(x,t)$ from equation (25), recalling the definition of the propagator G

$$\psi(x,t) = \int \mathrm{d}\eta\, G(x,t;\eta)\psi_0(\eta), \tag{27}$$

and comparing these two expressions, G follows at once: we go directly from the 'wavefunction' picture to the 'propagator' picture for our system. The propagator, for $\eta < 0$ and x outside the barrier, turns out to be (recall the definition (6) for G_0):

$$G(x,t;\eta) = G_0(x,t;\eta) - \frac{c^2}{4\hbar}\int_0^a \mathrm{d}\xi\, V(\xi)\mathcal{L}^{-1}\left\{\frac{\mathrm{e}^{-c(|x-\xi|+\xi-\eta)\sqrt{s}}}{\sqrt{s}(\sqrt{s}+\beta)}\right\} \tag{28}$$

holding for the transmission as well as for the reflection. Since [16]

$$\mathcal{L}^{-1}\left\{\frac{\mathrm{e}^{-\rho(\xi)\sqrt{s}}}{\sqrt{s}(\sqrt{s}+\beta)}\right\} = \mathrm{e}^{\beta^2 t+\rho(\xi)\beta}\,\mathrm{erfc}\left[\frac{\rho(\xi)}{2\sqrt{t}}+\beta\sqrt{t}\right], \qquad \rho(\xi) = c(|x-\xi|+\xi-\eta), \tag{29}$$

where (erfc) is the complementary error function [17], using equations (22) and (24) the final result is easily found in closed form. Let us now separate the two cases, according to whether $x > a$ (transmission) or $x < 0$ (reflection). The first is simpler, due to the fact that ξ disappears in the exponent ($\rho(\xi) = x - \eta$). For the sake of clarity, we repeat the steps (25)–(28):

$$\psi_T(x,s) = \frac{c}{2}\int \mathrm{d}\eta\left[\frac{\mathrm{e}^{-c(x-\eta)\sqrt{s}}}{\sqrt{s}} - \beta\frac{\mathrm{e}^{-c(x-\eta)\sqrt{s}}}{\sqrt{s}(\sqrt{s}+\beta)}\right]\psi_0(\eta) = \frac{c}{2}\int \mathrm{d}\eta\frac{\mathrm{e}^{-c(x-\eta)\sqrt{s}}}{\sqrt{s}+\beta}\psi_0(\eta) \tag{30}$$

giving [16]

$$G_T(x,t;\eta) = \frac{c}{2}\mathcal{L}^{-1}\left\{\frac{\mathrm{e}^{-c(x-\eta)\sqrt{s}}}{\sqrt{s}+\beta}\right\} = G_0(x,t;\eta) - \frac{c}{2}\beta\,\mathrm{e}^{\beta^2 t+c(x-\eta)\beta}\,\mathrm{erfc}\left[\frac{c(x-\eta)}{2\sqrt{t}}+\beta\sqrt{t}\right]. \tag{31}$$

For the reflection, $\rho(\xi) = 2\xi - x - \eta$, and one easily obtains

$$\begin{aligned} G_R(x,t;\eta) = G_0(x,t;\eta) - \frac{c^2}{4\hbar}\mathrm{e}^{\beta^2 t}\Bigg\{&U(\bar{a})\,\mathrm{e}^{c(2\bar{a}-x-\eta)\beta}\,\mathrm{erfc}\left(\frac{2\bar{a}-x-\eta}{2\sqrt{t}}+\beta\sqrt{t}\right)\\ &-\frac{\alpha}{4}\frac{d^2}{d\xi^2}\left[U(\xi)\,\mathrm{e}^{c(2\xi-x-\eta)\beta}\,\mathrm{erfc}\left(\frac{c(2\xi-x-\eta)}{2\sqrt{t}}+\beta\sqrt{t}\right)\right]_{\xi=\bar{a}}\Bigg\}. \end{aligned} \tag{32}$$

These equations generalize the results of [19] for a δ-like potential.

5. A more complicated system

An interesting application is now shown, observing that H_0 is not necessarily the free-particle Hamiltonian, but a general one, to which the potential V is added. Therefore, the following Hamiltonian can be considered:

$$H_0 = H_{\text{free}} + \begin{cases} V = \infty & \text{for} \quad x < 0 \\ V = 0 & \text{for} \quad x > 0 \end{cases} \tag{33}$$

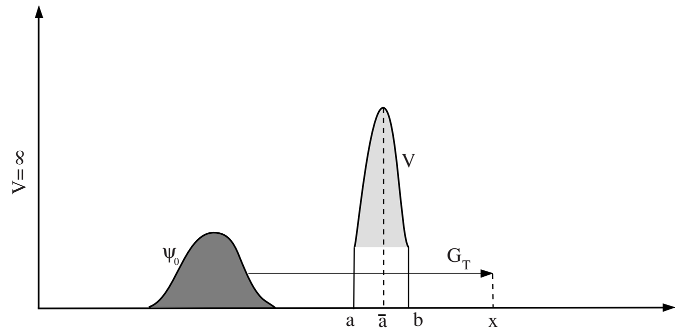

Figure 1. Transmission of a particle with initial wavefunction ψ_0 across the potential described in the text.

plus the potential

$$V(x,t) = V(x)[\theta(x-a) - \theta(x-b)], \qquad b > a > 0 \tag{34}$$

that appears in nuclear physics problems, as the virtual level theory of alpha decay [20]. This physical arrangement is shown in figure 1.

It is well known that in this case [13]

$$G_0(x,t;\eta) = G_{\text{free}}(x,t;\eta) - G_{\text{free}}(x,t;-\eta) \tag{35}$$

where G_{free} is defined by equation (6); the integral equation (11) now takes the form

$$\psi(x) + \lambda \int_a^b \mathrm{d}\xi [\mathrm{e}^{-k|x-\xi|} - \mathrm{e}^{-k|x+\xi|}] V(\xi)\psi(\xi) = \phi(x) \tag{36}$$

$\phi(x)$ being defined as

$$\phi(x) = \frac{c}{2}\int_{-\infty}^{\infty} \mathrm{d}\xi \; \psi_0(\xi) \frac{\mathrm{e}^{-c|x-\xi|\sqrt{s}} - \mathrm{e}^{-c|x+\xi|\sqrt{s}}}{\sqrt{s}}. \tag{37}$$

If $V(x)$ is the same as (22), where now $\bar{a} \in [a,b]$, and the range $(b-a)$ is short, the kernel can be considered again of the Pincherle–Goursat type. From the general theory [15] it is known that, following a procedure analogous to that used in the steps (12)–(17), the problem can be reduced to the solution of a (2×2) algebraic system; the calculation is quite simple and the final result reads

$$\psi(x) = \phi(x) - \frac{\lambda}{1+\lambda(\gamma-\tilde{\gamma})}\int_a^b \mathrm{d}\xi [\mathrm{e}^{-k|x-\xi|} - \mathrm{e}^{-k|x+\xi|}] V(\xi)\phi(\xi) \tag{38}$$

with

$$\gamma = \int_a^b \mathrm{d}x \; V(x) \qquad \text{given, as before, by} \qquad \left[U(x) + \frac{\alpha}{4}U^{(2)}(x)\right]_{x=\bar{a}}, \tag{39}$$

and

$$\tilde{\gamma} = \int_a^b \mathrm{d}x \, \mathrm{e}^{-2kx} V(x) = U(\bar{a})\,\mathrm{e}^{-2k\bar{a}} + \frac{\alpha}{4}\frac{d^2}{\mathrm{d}x^2}[U(x)\,\mathrm{e}^{-2kx}]_{x=\bar{a}} = \mathrm{e}^{-2k\bar{a}}(\gamma - k\mu + k^2\nu) \tag{40}$$

where

$$\mu = \alpha U'(\bar{a}), \qquad \nu = \alpha U(\bar{a}). \tag{41}$$

The propagator is therefore ($\eta < 0$; see equation (35) for G_0)

$$G(x,t;\eta) = G_0(x,t;\eta) - \frac{c^2}{4\hbar}\int_a^b \mathrm{d}\xi \; V(\xi)\mathcal{L}^{-1}\left\{\frac{\mathrm{e}^{-c(|x-\xi|+\xi-\eta)\sqrt{s}} - \mathrm{e}^{-c(x-\eta+2\xi)\sqrt{s}}}{\sqrt{s}(\sqrt{s}+\tilde{\beta})}\right\} \tag{42}$$

with $\tilde{\beta}$ given by (recall equation (26) for the definition of β)

$$\tilde{\beta} = \frac{c}{2}\frac{\gamma - \tilde{\gamma}}{\hbar} = \beta - \frac{c}{2}\frac{\tilde{\gamma}}{\hbar} \tag{43}$$

and the two cases of transmission and reflection are easily derived. The inversion is now difficult, since $\tilde{\gamma}$ depends on s (as $k = c\sqrt{s}$). One can, however, proceed in an approximate way: for example, if the barrier is far from the origin, the problem becomes more tractable. We consider in some detail this case, giving somewhat simple formulae for the transmission propagator G_T. First, we rewrite G_T in the form

$$G_T(x, t; \eta) = \frac{c}{2}\mathcal{L}^{-1}\left\{\frac{\mathrm{e}^{-c(x-\eta)\sqrt{s}}}{\sqrt{s}+\tilde{\beta}} - \frac{\mathrm{e}^{-c(x+\eta)\sqrt{s}}}{\sqrt{s}}\right\} \tag{44}$$

that can be easily obtained from equation (42) writing G_0 by its Laplace transform, observing that, in the transmission $(|x-\xi| + \xi - \eta) = (x - \eta)$, and recalling definitions (39) and (40). The path integration in the complex s-plane to perform the Laplace inversion is the line from $(\sigma_0 - \mathrm{i}\infty)$ to $(\sigma_0 + \mathrm{i}\infty)$, $\sigma_0 > 0$ [21]; it is easy to see that, on this line, $\mathrm{Re}\sqrt{s} > 0$, so that, if the barrier is far from the origin, one has $|\exp(-2\bar{a}c\sqrt{s})| \ll 1$. Therefore, a series expansion allows us to write, by means of equations (43) and (40),

$$(\sqrt{s}+\tilde{\beta})^{-1} \approx (\sqrt{s}+\beta)^{-1} + \frac{\mathrm{e}^{-2\bar{a}c\sqrt{s}}}{(\sqrt{s}+\beta)^2}\left(\beta - \frac{c^2\mu}{2\hbar}\sqrt{s} + \frac{c^3\nu}{2\hbar}s\right). \tag{45}$$

Substituting into equation (44), a short calculation leads to

$$\begin{aligned} G_T(x, t; \eta) \approx{}& \frac{c}{2}\mathcal{L}^{-1}\left\{\frac{\mathrm{e}^{-c(x-\eta)\sqrt{s}}}{\sqrt{s}+\beta} - \frac{\mathrm{e}^{-c(x+\eta)\sqrt{s}}}{\sqrt{s}}\right\} + \frac{c^3\mu}{4\hbar}\frac{\partial}{\partial\beta}\mathcal{L}^{-1}\left\{\frac{\sqrt{s}}{\sqrt{s}+\beta}\,\mathrm{e}^{-c(x-\eta+2\bar{a})\sqrt{s}}\right\} \\ &- \frac{c^4\nu}{4\hbar}\frac{\partial}{\partial\beta}\mathcal{L}^{-1}\left\{\frac{s}{\sqrt{s}+\beta}\,\mathrm{e}^{-c(x-\eta+2\bar{a})\sqrt{s}}\right\} - \frac{c}{2}\beta\frac{\partial}{\partial\beta}\mathcal{L}^{-1}\left\{\frac{\mathrm{e}^{-c(x-\eta+2\bar{a})\sqrt{s}}}{\sqrt{s}+\beta}\right\} \end{aligned} \tag{46}$$

and, since the inverse Laplace transforms are known [16], the result is

$$\begin{aligned} G_T(x, t; \eta) \approx{}& G_0(x, t; \eta) - \frac{c}{2}\beta\,\mathrm{e}^{\beta^2 t + c\beta(x-\eta)}\mathrm{erfc}\left[\frac{c(x-\eta)}{2\sqrt{t}} + \beta\sqrt{t}\right] \\ &+ \frac{c^3\mu}{4\hbar}\frac{\partial}{\partial\beta}\left\{\left[\frac{c(x-\eta+2\bar{a})}{2\sqrt{t}} - \beta\sqrt{t}\right]\frac{\mathrm{e}^{-c^2(x-\eta+2\bar{a})^2/4t}}{\sqrt{\pi}t}\right. \\ &\left.+ \beta^2\,\mathrm{e}^{\beta^2 t + c\beta(x-\eta+2\bar{a})}\mathrm{erfc}\left[\frac{c(x-\eta+2\bar{a})}{2\sqrt{t}} + \beta\sqrt{t}\right]\right\} \\ &- \frac{c^4\nu}{4\hbar}\frac{\partial}{\partial\beta}\left\{\left[\frac{c^2(x-\eta+2\bar{a})^2}{4t} - \frac{1}{2}c\beta(x-\eta+2\bar{a}) + \beta^2 t - \frac{1}{2}\right]\frac{\mathrm{e}^{-c^2(x-\eta+2\bar{a})^2/4t}}{\sqrt{\pi}t^{3/2}}\right. \\ &\left.- \beta^3\,\mathrm{e}^{\beta^2 t + c\beta(x-\eta+2\bar{a})}\,\mathrm{erfc}\left[\frac{c(x-\eta+2\bar{a})}{2\sqrt{t}} + \beta\sqrt{t}\right]\right\} \\ &- \frac{c}{2}\beta\frac{\partial}{\partial\beta}\left\{\frac{\mathrm{e}^{-c^2(x-\eta+2\bar{a})^2/4t}}{\sqrt{\pi t}} - \beta\,\mathrm{e}^{\beta^2 t + c\beta(x-\eta+2\bar{a})}\mathrm{erfc}\left[\frac{c(x-\eta+2\bar{a})}{2\sqrt{t}} + \beta\sqrt{t}\right]\right\}. \end{aligned} \tag{47}$$

Recall that our formulae are written for imaginary time; to obtain the propagator in real time, the change $t \to \mathrm{i}t$ must be performed. Therefore, although this expression can seem a quite standard one, one has really to deal with the (erfc) of a complex argument, and the computation is not elementary [17].

We eventually observe that also three-dimensional problems can be afforded, using an expansion in spherical harmonics [22]; this method applies to short-range potentials, in a particularly simple way for S-states.

6. Conclusions

The Schrödinger equation in integral form has been used in a one-dimensional model to find the propagator for narrow potential barriers. Applying some methods of integral equation theory, and assuming that the kernel can be considered everywhere separable (approximation holding in the case of a narrow barrier), a closed form for the propagator is obtained. The more difficult situation of a particle lying in a half-space and interacting with a potential barrier is also considered and solved in an approximate way when the barrier is far from the origin. The simple expressions so obtained, especially in the transmission case, show that the theory based on the integral equation is far more useful and important than the mere mathematical equivalence with the ordinary method would have suggested. It is therefore possible that this approach, not widely used up to now, can lead to other interesting results.

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